ON THE CONTACT OF QUASILINEAR OBJECTS IN THE REGULAR CASE

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We consider the game problem of the contact of the motions of controlled objects [1-6] whose behavior is described by quasilinear differential equations. Under instantaneous constraints on the values of the controls we show that in the regular case an inhibiting extremal strategy [5, 6] ensures contact no later than at the instant of absorption.

1. Suppose that the pursuing motion $y[t]$ and the pursued motion $z[t]$ of the controlled objects are described by the equations

$$y^{\cdot} = A^{(1)}(t)\, y + B^{(1)}(t)\, u + \lambda f^{(1)}(y, t) \tag{1.1}$$

$$z^{\cdot} = A^{(2)}(t)\, z + B^{(2)}(t)\, v + \lambda f^{(2)}(z, t) \tag{1.2}$$

where y and z are, respectively, $n^{(1)}$- and $n^{(2)}$ -dimensional phase vectors; u and v are $r^{(1)}$- and $r^{(2)}$ -dimensional control vectors; $A^{(j)}$ and $B^{(j)}$ are matrices of appropriate dimensions; $f^{(1)}(y, t)$ and $f^{(2)}(z, t)$ are vector-valued functions continuously differentiable in t and twice continuously differentiable in y and z for $y \in \Gamma_1$ and $z \in \Gamma_2$, where Γ_1 and Γ_2 are some closed bounded regions; λ $(\lambda > 0)$ is a small parameter. Constraints are imposed on the controls u and v ; $u[t] \in U$, $v[t] \in V$, where the sets U and V of vectors u and v are described by the inequalities

$$\|u[t]\| \leqslant \mu, \qquad \|v[t]\| \leqslant \nu \quad (\mu, \nu = \text{const}) \tag{1.3}$$

Here and everywhere in what follows the symbol $\|x\|$ denotes the Euclidean norm of the vector x. We shall say that a contact of the motions $y[t]$ and $z[t]$, takes place at the instant $t = \vartheta \geqslant t_0$ if the equality

$$\{y[\vartheta]\}_m = \{z[\vartheta]\}_m$$

is first fulfilled at $t = \vartheta$; the symbol $\{x\}_m$ denotes the vector made up of the first m components of vector x. We derive below the proof of the inhibiting extremal strategy [5, 6] in the case of the quasilinear objects (1.1) and (1.2) when $\lambda \leqslant \lambda_0$, where λ_0 is a sufficinetly small positive number.

2. We cite certain facts which are used subsequently. For this we shall assume that the following conditions are fulfilled.

Condition 2.1 When $\lambda = 0$, the motions of systems (1.1) and (1.2), generated by all possible controls subject to constraints (1.3), with the initial conditions

$y_0 \in \Gamma_1^\circ \subset \Gamma_1$ and $z_0 \in \Gamma_2^\circ \subset \Gamma_2$, are wholly contained in the regions Γ_1 and Γ_2.

Let $Y^\circ[\vartheta, \tau]$ and $Z^\circ[\vartheta, \tau]$ be the fundamental matrices of Eqs. (1.1) and (1.2) when $\lambda = 0$, $u \equiv 0$, $v \equiv 0$. We denote

$$\xi_1(\tau) = \| l'\{Y^\circ[\vartheta, \tau] B^{(1)}(\tau)\}_m \|, \qquad \xi_2(\tau) = \| l'\{Z^\circ[\vartheta, \tau] B^{(2)}(\tau)\}_m \| \qquad (2.1)$$

Condition 2.2. Whatever be the m-dimensional unit vector l ($\|l\| = 1$), the functions $\xi_1(\tau)$, $\xi_2(\tau)$ in (2.1) can vanish only at a finite number of points $\tau_k^{(1)}$ and $\tau_r^{(2)}$ of the interval $[t, \vartheta]$, where ϑ is any finite number greater than t, and

$$\left| \frac{d\xi_1}{d\tau} \Big|_{\tau=\tau_k^{(1)}} \right| \geqslant k_1 > 0, \left| \frac{d\xi_2}{d\tau} \Big|_{\tau=\tau_r^{(2)}} \right| \geqslant k_2 > 0 \quad (k_1, k_2 = \text{const})$$

Let us consider the controls $u^\circ(\tau)$ and $v^\circ(\tau)$ ($t \leqslant \tau \leqslant \vartheta$), which are solutions of the problems

$$\rho^{(1)}[l, \vartheta, t, y, \lambda] = \max_u l'\{y(\vartheta; u)\}_m = l'\{y^\circ(\vartheta; l, \vartheta, t, y, \lambda)\}_m \quad (u \in U)$$
$$\rho^{(2)}[l, \vartheta, t, z, \lambda] = \max_v l'\{z(\vartheta; v)\}_m = l'\{z^\circ(\vartheta; l, \vartheta, t, z, \lambda)\}_m \quad (v \in V) \qquad (2.2)$$

where $y(\tau; u)$ and $z(\tau; v)$ are the motions of systems (1.1) and (1.2), generated by certain controls $u(\tau)$ and $v(\tau)$ subject to constraints (1.3), under the initial condition $\tau = t$, $y(t; u) = y$, $z(t; v) = z$, and l is an arbitrary unit vector. The motions $y^\circ(\tau; l, \vartheta, t, y, \lambda)$ and $z^\circ(\tau; l, \vartheta, t, z, \lambda)$, satisfying equalities (2.2), are generated by the controls $u^\circ(\tau)$ and $v^\circ(\tau)$ which for each τ from the interval $[t, \vartheta]$ are determined by the maximum condition

$$l'\{Y[\vartheta, \tau; l, \vartheta, t, y, \lambda] B^{(1)}(\tau)\}_m u^\circ(\tau) =$$
$$= \max_u l'\{Y[\vartheta, \tau; l, \vartheta, t, y, \lambda] B^{(1)}(\tau)\}_m u \qquad (u \in U)$$
$$l'\{Z[\vartheta, \tau; l, \vartheta, t, z, \lambda] B^{(2)}(\tau)\}_m v^\circ(\tau) =$$
$$= \max_v l'\{Z[\vartheta, \tau; l, \vartheta, t, z, \lambda] B^{(2)}(\tau)\}_m v \qquad (v \in V)$$

where Y and Z are the fundamental matrices of the systems of equations in variations

$$d\delta y/d\tau = A^{(1)\circ}(\tau; l, \vartheta, t\ y, \lambda)\,\delta y, \qquad d\delta z/d\tau = A^{(2)\circ}(\tau; l, \vartheta, t, z, \lambda)\,\delta z$$

set up for Eqs. (1.1) and (1.2) respectively, along the motions $y^\circ(\tau; l, \vartheta, t, y, \lambda)$ and $z^\circ(\tau, l, \vartheta, t, z, \lambda)$. From the results of [7, 8] and from condition (2.2) it follows that when $\lambda \leqslant \lambda_0$ the controls $u^\circ(\tau)$ and $v^\circ(\tau)$ are uniquely defined for each vector l and consequently, for each vector l there exist unique points $y^\circ(\vartheta; l, \vartheta, t, y, \lambda)$ and $z^\circ(\vartheta; l, \vartheta, t, z, \lambda)$ satisfying equalities (2.2).

We introduce into consideration the quantity

$$\varepsilon^\circ(\vartheta, t, y, z, \lambda) = \max_l \{\rho^{(2)}[l, \vartheta, t, z, \lambda] -$$
$$- \rho^{(1)}[l, \vartheta, t, y, \lambda]\} \qquad (\|l\| = 1) \qquad (2.3)$$

By the definition of the functions $\rho^{(1)}$ and $\rho^{(2)}$ in (2.2), when $t = \vartheta$ the quantity ε° in (2.3) equals the distance between the points $\{y[\vartheta]\}_m$ and $\{z[\vartheta]\}_m$.

Definition 2.1. We shall say that the regular case [6] obtains if for each position $\{t, y, z\}$ for which $0 < \varepsilon^\circ(\vartheta, t, y, z, \lambda) < \alpha$ (α is a sufficiently small positive number) the maximum in the right hand side of equality (2.3) is attained on a unique vector $l^\circ = l^\circ(\vartheta, t, y, z, \lambda)$.

Definition 2.2. The smallest value of $\vartheta \geqslant t$, for which the equality

$$\max_l \{\rho^{(2)}[l, \vartheta, t, z, \lambda] - \rho^{(1)}[l, \vartheta, t, y, \lambda]\} = 0 \qquad (\|l\| = 1) \tag{2.4}$$

is fulfilled will be called the absorption instant $\vartheta^\circ = \vartheta^\circ(t, y, z, \lambda)$ of the process $z[t]$ in (1.2) by the process $y[t]$ in (1.1).

Let us now assume that on a certain interval $[t_*, \vartheta]$ (ϑ is fixed) the players employ admissible strategies [6]. Here, at each instant $t \in [t_*, \vartheta]$ only those position $\{t, y, z\}$ ($y = y[t], z = z[t]$), are realized for which $\varepsilon^\circ(\vartheta, t, y, z, \lambda) > 0$. Let us examine the function $\varepsilon^\circ[t] = \varepsilon^\circ(\vartheta, t, y[t], z[t], \lambda)$.

Theorem 2.1. Let Conditions 2.1 and 2.2 be fulfilled. Then in the regular case, when $\lambda \leqslant \lambda_0$ the function $\varepsilon^\circ[t]$ is absolutely continuous and for almost all $t \in [t_*, \vartheta]$,

$$\begin{aligned} d\varepsilon^\circ[t]/dt = {} & \max_u l^{\circ\prime}\{Y[\vartheta, t; l^\circ, \vartheta, t, y, \lambda]B^{(1)}(t)\}_m u - \\ & - l^{\circ\prime}\{Y[\vartheta, t; l^\circ, \vartheta, t, y, \lambda]B^{(1)}(t)\}_m u[t] - \qquad (u \in U) \\ & - \max_v l^{\circ\prime}\{Z[\vartheta, t; l^\circ, \vartheta, t, z, \lambda]B^{(2)}(t)\}_m v + l^{\circ\prime}\{Z[\vartheta, t; l^\circ, \vartheta, t, z, \\ & \lambda]B^{(2)}(t)\}_m v[t] \qquad (v \in V) \end{aligned} \tag{2.5}$$

3. Let us describe the construction of an approximate inhibiting extremal strategy [4-6]. Let $\vartheta^\circ = \vartheta^\circ(t_0, y_0, z_0, \lambda)$ be the absorption instant corresponding to the initial position $\{t_0, y_0, z_0\}$. We divide up the interval $[t_0, \vartheta^\circ]$ into the semi-intervals $[\tau_k, \tau_{k+1})$ ($k = 0, 1, 2, \ldots; \tau_0 = t_0$) of equal length $\Delta = \tau_{k+1} - \tau_k$. We shall assume that on each of these semi-intervals the control $u_{e\Delta}$ is constant and is constructed in the form

$$u_{e\Delta}[t] = u_\Delta[\tau_k, y_\Delta[\tau_k], z[\tau_k], \vartheta_\Delta[\tau_k], \lambda] = u_{e\Delta}[\tau_k]$$
$$(\tau_k \leqslant t < \tau_{k+1})$$

where $\vartheta_\Delta[t]$ is some auxiliary variable which also is taken as being constant on each semi-interval $[\tau_k, \tau_{k+1})$, i.e.,

$$\vartheta_\Delta[t] = \vartheta_\Delta[\tau_k] \qquad (\tau_k \leqslant t < \tau_{k+1})$$

Here the values of $\vartheta_\Delta[\tau_k]$ are determined as follows. When $t = t_0$ we assume that $\vartheta_\Delta[t_0] = \vartheta^\circ$. Suppose that the position $\{\tau_k, y_\Delta[\tau_k], z[\tau_k]\}$ is realized at $t = \tau_k$ and that this position corresponds to the absorption instant $\vartheta_{\Delta k}^\circ = \vartheta^\circ(\tau_k, y_\Delta[\tau_k], z[\tau_k], \lambda)$. For $\tau_k \leqslant t < \tau_{k+1}$ we shall define the quantity $\vartheta_\Delta[t]$ thus

$$\vartheta_\Delta[t] = \begin{cases} \vartheta_{\Delta k}^\circ & \text{for } \vartheta_{\Delta k}^\circ \leqslant \vartheta_\Delta[\tau_{k-1}] \\ \vartheta_\Delta[\tau_{k-1}] & \text{for } \vartheta_{\Delta k}^\circ > \vartheta_\Delta[\tau_{k-1}] \end{cases}$$

We now describe a method for choosing the values of $u_{e\Delta}[\tau_k]$. If $\vartheta_\Delta[\tau_k] = \vartheta_{\Delta k}^\circ$, then we assume

$$u_{e\Delta}[\tau_k] \in U \tag{3.1}$$

i.e. $u_{e\Delta}[\tau_k]$ is any value of the control u from the set U. If, however $\vartheta_\Delta[\tau_k] <$

$< \vartheta_{\Delta k}$, then

for $\|l_{\Delta k}^{\circ\prime}\{YB^{(1)}\}_m\| \neq 0$ (3.2)

$$u_{e\Delta}[\tau_k] = \mu \frac{(l_{\Delta k}^{\circ\prime}\{Y[\vartheta_\Delta[\tau_{k-1}], \tau_k; l_{\Delta k}^\circ, \vartheta_\Delta[\tau_{k-1}], \tau_k, y_\Delta[\tau_k], \lambda] B^{(1)}(\tau_k)\}_m)'}{\|l_{\Delta k}^{\circ\prime}\{Y[\vartheta_\Delta[\tau_{k-1}], \tau_k; l_{\Delta k}^\circ, \vartheta_\Delta[\tau_{k-1}], \tau_k, y_\Delta[\tau_k], \lambda] B^{(1)}(\tau_k)\}_m\|}$$

for $\|l_{\Delta k}^\circ\{YB^{(1)}\}_m\| = 0, \quad u_{e\Delta}[\tau_k] \in U$ (3.3)

Here $l^\circ_{\Delta k} = l^\circ \quad (\vartheta_\Delta[\tau_{k-1}], \tau_k, y_\Delta[\tau_k], z[\tau_k], \lambda)$ is the vector on which the maximum is attained in the right hand side of equality (2.3) when $t = \tau_k$ and $\vartheta = = \vartheta_\Delta[\tau_{k-1}]$.

The following assertion is valid.

Theorem 3.1. Suppose that Conditions 2.1 and 2.2 are fulfilled and that the regular case holds, then when $\lambda \leq \lambda_0$ for any arbitrarily small number $\eta > 0$ we can find a number $\Delta^\circ > 0$ such that for all $0 < \Delta \leq \Delta^\circ$, when the pursuer has chosen an approximate inhibiting extremal strategy and the pursued has chosen any admissible strategy, we can find an instant $\vartheta^* \leq \vartheta^\circ(t_0, y_0, z_0, \lambda)$ at which the inequality

$$\|\{y_\Delta[\vartheta^*]\}_m - \{z[\vartheta^*]\}_m\| \leq \eta$$

is fulfilled if only the initial position $\{t_0, y_0, z_0\}$ $(y_0 \in \Gamma_1^\circ, z_0 \in \Gamma_2^\circ)$ is such that the absorption instant ϑ° exists.

Proof. At first we assume that a position $\{\tau_k, y_\Delta[\tau_k], z[\tau_k]\}$ is realized at $t = \tau_k$ such that $\vartheta_{\Delta k}[\tau_k] = \vartheta^\circ_{\Delta k}$, then by definition, $\varepsilon_\Delta^\circ[\tau_k] = 0$. From the continuity of the functions $\rho^{(1)}$ and $\rho^{(2)}$ in (2.2) it follows that the inequality

$$\varepsilon_\Delta^\circ[\tau_{k+1}] \leq \delta(\Delta) \qquad (y_\Delta[\tau_k] \in \Gamma_1, z[\tau_k] \in \Gamma_2)$$

is valid for $t = \tau_{k+1}$, where $\lim_{\Delta \to 0} \delta(\Delta) = 0$ uniformly with respect to $\lambda \leq \lambda_0$.

We now assume that when $t = \tau_k$ a position is realized such that $\vartheta_{\Delta k}^\circ > \vartheta_\Delta[\tau_{k-1}]$, then $\varepsilon_\Delta^\circ[\tau_k] > 0$. From (2.5) and (3.2) - (3.3) we get that

$$\varepsilon_\Delta^\circ[\tau_{k+1}] - \varepsilon_\Delta^\circ[\tau_k] \leq o(\Delta)$$

Here

$$\lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0 \qquad (y_\Delta[\tau_k] \in \Gamma_1, \ z[\tau_k] \in \Gamma_2)$$

uniformly with respect to $\lambda \leq \lambda_0$. Consequently, the inequality

$$\varepsilon_\Delta^\circ[\vartheta^\circ] \leq \frac{\vartheta^\circ - t_0}{\Delta} o(\Delta) + \delta(\Delta) \qquad (3.4)$$

is valid. From (3.4) we obtain that for any arbitrarily small number $\eta > 0$ we can find a number $\Delta^\circ > 0$ such that the inequality

$$\varepsilon_\Delta^\circ[\vartheta^\circ] < \eta \qquad (3.5)$$

is fulfilled for all $0 < \Delta \leq \Delta^\circ$. But when $t = \vartheta^\circ$ the quantity $\varepsilon_\Delta^\circ[\vartheta^\circ]$ is, by definition, the distance between the points $\{y_\Delta[\vartheta^\circ]\}_m$ and $\{z[\vartheta^\circ]\}_m$.

Therefore, the validity of the theorem's assertion follows from (3.5).

4. The discussions of the preceding Section permit us to give the following formal definition of an inhibiting extremal strategy U_e^* [5, 6]. Let us consider the $(n^{(1)} + n^{(2)} + 1)$-dimensional phase space W consisting of the elements $\{y, z, \vartheta\}$, where ϑ is a scalar variable and $\vartheta \geqslant 0$. We separate the space W into the parts W_0 and W_ε. The set W_0 is the collection of those and only those points $\{y, z, \vartheta\}$ for which $\vartheta \geqslant \vartheta^\circ(t, y, z, \lambda)$, while W_ε, to the contrary, is the collection of those points $\{y, z, \vartheta\}$ for which $\vartheta < \vartheta^\circ(t, y, z, \lambda)$. We define the strategy U_e^* as the sets $U_e(t, y, z, \vartheta, \lambda)$ depending on the variables $t, y, z, \vartheta, \lambda$ in the following manner:

$$U_e(t, y, z, \vartheta, \lambda) = U, \quad \text{if} \quad \{y, z, \vartheta\} \in W_0$$

$$U_e(t, y, z, \vartheta, \lambda) = U_e^{(\varepsilon)}, \quad \text{if} \quad \{y, z, \vartheta\} \in W_\varepsilon$$

Moreover, the set $U_e^{(\varepsilon)}(t, y, z, \vartheta, \lambda)$ is the collection of all those vectors $u_e[t]$ which satisfy the maximum condition

$$l^{\circ\prime}\{Y[\vartheta, t; l^\circ, \vartheta, t, y, \lambda] B^{(1)}(t)\}_m u_e[t] =$$
$$= \max_u l^{\circ\prime}\{Y[\vartheta, t; l^\circ, \vartheta, t, y, \lambda] B^{(1)}(t)\}_m u \qquad (u \in U)$$

where l° is the vector on which the maximum in the right hand side of equality (2.3) is reached. Here, at the instant t when $\|l^{\circ\prime}\{YB^{(1)}\}_m\| \neq 0$ the sets $U_e^{(\varepsilon)}$ consist of the single point

$$u_e[t] = \mu \frac{(l^{\circ\prime}\{YB^{(1)}\}_m)'}{\|l^{\circ\prime}\{YB^{(1)}\}_m\|}$$

whereas at the instant t when $\|l^{\circ\prime}\{YB^{(1)}\}_m\| = 0$ we assume that $U_e^{(\varepsilon)} = U$.

Let us now augment the system (1.1) and (1.2) by relations which define the change in the variable $\vartheta[t]$ with the course of time t. We shall take it [5, 6] that the function $\vartheta[t]$ is right-continuous and satisfies the conditions

$$(d\vartheta/dt)^+ \leqslant 0, \quad \text{if} \quad \{y, z, \vartheta\} \in W_0$$
$$d\vartheta/dt = 0, \quad \text{if} \quad \{y, z, \vartheta\} \in W_\varepsilon \tag{4.1}$$

The symbol $(d\vartheta/dt)^+$ denotes the right upper derivative.

As the discrete scheme described in Sect. 3 goes to the limit as $\Delta \to 0$, we can show [5 - 9] that the inhibiting extremal strategy U_e^* ensures the existence of the solution $\{y[t], z[t], \vartheta[t]\}$ of Eqs. (1.1), (1.2), (4.1), satisfying the initial condition

$$\{y[t_0], z[t_0], \vartheta[t_0]\} \in W_0 \qquad (\vartheta[t_0] = \vartheta^\circ)$$

and, for all $t \geqslant t_0$ until contact is effected, contained in the set W_0. Consequently, the following assertion is valid.

Theorem 4.1. Suppose that Conditions 2.1 and 2.2 are fulfilled and that the regular case obtains, then when $\lambda \leqslant \lambda_0$ the inhibiting extremal strategy U^*_e ensures the contact of the motions $y[t]$ and $z[t]$ not later than at the absorption instant $\vartheta^\circ = \vartheta(t_0, y[t_0], z[t_0], \lambda)$ no matter what the admissible realization $v[t]$ of control v is, if only the initial position $\{t_0, y[t_0], z[t_0]\}$ $(y[t_0] \in \Gamma_1^\circ, z[t_0] \in \Gamma_2^\circ)$ is such that the absorption instant exists.

Note 4.1. All the preceding assertions easily carry over to a more general case of the game problem of the contact of quasilinear objects. Let P be a given convex, closed and bounded set consisting of m-dimensional vectors p. We say [6] that the point $\{z[t]\}_m$ is contained in the region of influence $M(\{y[t]\}_m)$ of the point $\{y[t]\}_m$ if and only if the vector $p = \{z[t]\}_m - \{y[t]\}_m$ is contained in set P. The instant $t = \vartheta \geqslant t_0$ when the point $\{z[\vartheta]\}_m$ first falls into the region of influence $M(\{y[\vartheta]\}_m)$ of the point $\{y[\vartheta]\}_m$ is called the instant of contact of the motions $y[t]$ of (1.1) and $z[t]$ of (1.2). In the case of such a pursuit problem (when Conditions 2.1 and 2.2 are fulfilled) we can also make statements similar to Theorems 3.1 and 4.1.

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